ASYMPTOTIC ANALYSIS OF A SOLUTION OF THE PROBLEM OF THE THEORY OF ELASTICITY FOR A SHALLOW CONE

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The axisymmetric problem of the theory of elasticity is considered for a body bounded by two spherical and two conical surfaces. The results of [1, 2] are used to perform an asymptotic analysis of the stress-strain state of the shell. Methods developed in [3, 4] are used to reduce the boundary value problem to infinite systems.

1. We consider an elastic conical shell, using the spherical coordinate system

$$r, \quad \theta, \quad \varphi \quad (r_1 \leqslant r \leqslant r_2, \quad \theta_1 \leqslant \theta \leqslant \theta_2, \quad 0 \leqslant \varphi \leqslant 2\pi)$$

and assume that the conditions

$$\sigma_\theta = 0, \qquad \tau_{r\theta} = 0 \tag{1.1}$$

hold on the conical boundaries $(\theta = \theta_1, \ \theta = \theta_2)$.

Using the results of [1, 2], we apply the method of homogeneous solutions to express the stresses and displacements in the form

$$u_r = u_r^\circ + r^{-1/2} \sum_{k=1}^{\infty} C_k r^{z_k} U_{rk}, \qquad u_\theta = u_\theta^\circ + r^{-1/2} \sum_{k=1}^{\infty} C_k r^{z_k} U_{\theta k} \tag{1.2}$$

$$\sigma_r = \sigma_r^\circ + 2Gr^{-3/2} \sum_{k=1}^{\infty} C_k r^{z_k} Q_{rk}$$

$$\sigma_\theta = \sigma_\theta^\circ + 2Gr^{-3/2} \sum_{k=1}^{\infty} C_k r^{z_k} Q_{\theta k}, \qquad \sigma_\varphi = \sigma_\varphi^\circ + 2Gr^{-3/2} \sum_{k=1}^{\infty} C_k r^{z_k} Q_{\varphi k} \tag{1.3}$$

$$\tau_{r\theta} = \tau_{r\theta}^\circ + 2Gr^{-3/2} \sum_{k=1}^{\infty} C_k r^{z_k} T_k$$

where

$$u_r^\circ = C_0 r^{-1} [4 (1 - \nu) \cos \theta - (1 - 2\nu) (\cos \theta_1 + \cos \theta_2)] - A \cos \theta$$
$$u_\theta^\circ = C_0 r^{-1} [(3 - 4\nu) \sin \theta - (1 - 2\nu) (1 + \cos \theta_1 \cos \theta_2) \csc \theta + \tag{1.4}$$
$$+ (1 - 2\nu) (\cos \theta_1 + \cos \theta_2) \operatorname{ctg} \theta] + A \sin \theta$$

$$\sigma_r^\circ = 2GC_0 r^{-2} [2 (2 - \nu) \cos \theta - (1 - 2\nu) (\cos \theta_1 + \cos \theta_2)]$$
$$\sigma_\theta^\circ = -2 (1 - 2\nu) GC_0 r^{-2} [\cos \theta - (1 + \cos \theta_1 \cos \theta_2) \operatorname{ctg} \theta \csc \theta +$$
$$+ (\cos \theta_1 + \cos \theta_2) \operatorname{ctg}^2 \theta]$$
$$\sigma_\varphi^\circ = -2 (1 - 2\nu) GC_0 r^{-2} [\cos \theta + (1 + \cos \theta_1 \cos \theta_2) \operatorname{ctg} \theta \csc \theta +$$
$$+ (\cos \theta_1 + \cos \theta_2) (1 + \operatorname{ctg}^2 \theta)] \tag{1.5}$$
$$\tau_{r\theta}^\circ = -2 (1 - 2\nu) GC_0 r^{-2} [\sin \theta - (1 + \cos \theta_1 \cos \theta_2) \csc \theta + (\cos \theta_1 + \cos \theta_2) \operatorname{ctg} \theta$$

$$
\begin{aligned}
U_{rk} &= (z_k - {}^1\!/_2)(z_k + 4\nu - {}^7\!/_2)\,\psi_1(\theta, z_k) - (z_k + {}^1\!/_2)\,\psi_2(\theta, z_k) \\
U_{\theta k} &= (z_k - 4\nu + {}^7\!/_2)\,\frac{d\psi_1(\theta, z_k)}{d\theta} - \frac{d\psi_2(\theta, z_k)}{d\theta} \\
Q_{rk} &= (z_k - {}^1\!/_2)(z_k^2 - 4z_k + {}^7\!/_4 + 2\nu)\,\psi_1(\theta, z_k) - (z_k^2 - {}^1\!/_4)\,\psi_2(\theta, z_k) \\
Q_{\theta k} &= -(z_k - {}^1\!/_2)(z_k^2 + z_k + 2\nu - {}^7\!/_4)\,\psi_1(\theta, z_k) - (z_k + {}^7\!/_4 - 4\nu)\,d\psi_1/d\theta + \\
&\quad + (z_k + {}^1\!/_2)^2\,\psi_2(\theta, z_k) + \operatorname{ctg}\theta\, d\psi_2/d\theta \\
Q_{\varphi k} &= (z_k - {}^1\!/_2)(z_k - {}^7\!/_2 + 4\nu)\,\psi_1(\theta, z_k) + (z_k + {}^7\!/_4 - 4\nu)\operatorname{ctg}\theta\, d\varphi_1/d\theta - \\
&\quad - (z_k + {}^1\!/_2)\,\psi_2(\theta, z_k) - \operatorname{ctg}\theta\, d\psi_2/d\theta \\
T_k &= (z_k^2 - z_k + 2\nu - {}^7\!/_4)\,d\psi_1/d\theta - (z_k - {}^1\!/_2)\,d\psi_2/d\theta
\end{aligned}
\qquad (1.6)
$$

$$
\begin{aligned}
\psi_1(\theta, z) &= (z - {}^1\!/_2)(z^2 - {}^1\!/_4)\,\varphi_1(-z)\,[\csc\theta_2 D_{z-{}^3\!/_2}(\theta, \theta_1) - \\
&\quad - D^{(1,0)}_{z+{}^1\!/_2}(\theta_1, \theta_2)\, D^{(0,1)}_{z-{}^3\!/_2}(\theta, \theta_2)] + \\
&\quad + 4(1 - \nu)\, z\,(z - {}^1\!/_2)\operatorname{ctg}\theta_2 D^{(1,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2)\, D^{(0,1)}_{z-{}^3\!/_2}(\theta, \theta_2) + \\
&\quad + (z - {}^1\!/_2)^2\,\varphi_1(z)\, D^{(1,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2)\, D^{(0,0)}_{z-{}^3\!/_2}(\theta, \theta_2)
\end{aligned}
\qquad (1.7)
$$

$$
\begin{aligned}
\psi_2(\theta, z) &= -(z - {}^1\!/_2)^2\,\varphi_1(z)\,\varphi_1(-z)\,[\csc\theta_2 D_{z+{}^1\!/_2}(\theta, \theta_2) - \\
&\quad - D^{(1,0)}_{z-{}^3\!/_2}(\theta_1, \theta_2) D^{(0,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2)] + \\
&\quad + 4(1 - \nu)\, z\,\varphi_1(-z)\operatorname{ctg}\theta_2 D^{(1,1)}_{z-{}^3\!/_2}(\theta_1, \theta_2)\, D^{(0,1)}_{z+{}^1\!/_2}(\theta, \theta_2) + \\
&\quad + (z + {}^1\!/_2)^2\,\varphi_1^2(-z_k)\, D^{(1,1)}_{z-{}^3\!/_2}(\theta_1, \theta_2)\, D^{(0,1)}_{z+{}^1\!/_2}(\theta, \theta_2)
\end{aligned}
\qquad (1.8)
$$

$$
D_l^{(s,l)}(\varphi_1\psi) = P_l^{(s)}(\cos\varphi)\, Q_l^{(l)}(\cos\psi) - P_l^{(s)}(\cos\psi)\, Q_l^{(l)}(\cos\varphi) \qquad (1.9)
$$

In the formulas (1.2) - (1.9) $P_l^{(s)}(\cos\varphi)$ and $Q_l^{(l)}(\cos\varphi)$ denote the Legendre functions of the first and second kind, respectively, $\varphi_1(z) = z^2 + z + 2\nu - {}^7\!/_4$, G and ν are elastic constants, A, C_0 and C_k are arbitrary constants and z_k are the complex zeros of the function

$$
\begin{aligned}
\Delta(z) &= -(z - {}^1\!/_2)^4\varphi_1^2(z) D^{(0,0)}_{z-{}^3\!/_2}(\theta_1, \theta_2)\, D^{(1,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2) - \\
&\quad - (z + {}^1\!/_2)^4\varphi_1^2(-z)\, D^{(0,0)}_{z+{}^1\!/_2}(\theta_1, \theta_2)\, D^{(1,1)}_{z-{}^3\!/_2}(\theta_1, \theta_2) + \\
&\quad + 4(1 - \nu)\, z\,(z - {}^1\!/_2)^2\,\varphi_1(z)\, D^{(1,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2)\,[\operatorname{ctg}\theta_2 D^{(0,1)}_{z-{}^3\!/_2}(\theta_1, \theta_2) + \\
&\quad + \operatorname{ctg}\theta_1 D^{(1,0)}_{z-{}^3\!/_2}(\theta_1, \theta_2)] - 4(1 - \nu)\, z\,(z + {}^1\!/_2)^2\,\varphi_1(-z)\, D^{(1,1)}_{z-{}^3\!/_2}(\theta_1, \theta_2) \times \\
&\quad \times [\operatorname{ctg}\theta_2 D^{(0,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2) + \operatorname{ctg}\theta_1\, D^{(1,0)}_{z+{}^1\!/_2}(\theta_1, \theta_2)] + \\
&\quad + 16(1 - \nu^2)\, z^2 \operatorname{ctg}\theta_1 \operatorname{ctg}\theta_2\, D^{(1,1)}_{z-{}^3\!/_2}\, D^{(1,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2) - \\
&\quad - (z^2 - {}^1\!/_4)^2\,\varphi_1(z)\,\varphi_1(-z)\,[D^{(0,1)}_{z-{}^3\!/_2}(\theta_1, \theta_2)\, D^{(1,0)}_{z+{}^1\!/_2}(\theta_1, \theta_2) + \\
&\quad + D^{(1,0)}_{z-{}^3\!/_2}(\theta_1, \theta_2)\, D^{(0,1)}_{z+{}^1\!/_2}(\theta_1, \theta_2)] - 2(z^2 - {}^1\!/_4)\,\varphi_1(z)\,\varphi_1(-z)\csc\theta_1\csc\theta_2
\end{aligned}
\qquad (1.10)
$$

The terms u_r°, u_θ° and $\overset{\circ}{\tau}_{r\theta}$ appearing in (1.2) and (1.3) correspond to the real zeros $z_{01} = -0.5$ and $z_{02} = 0.5$ of the function $\Delta(z)$.

2. We now turn our attention to the pattern of the state of stress described by the homogeneous solutions (1.2) - (1.8).

We first consider the relationship connecting the homogeneous solutions with the principal stress vector P acting at the cross section $r = \text{const}$. We have

$$
P = 2\pi r^2 \int_{\theta_1}^{\theta_2} (\sigma_r \cos\theta - \tau_{r\theta}\sin\theta)\sin\theta\, d\theta \qquad (2.1)
$$

Inserting (1.3) into (2.1) we obtain

$$P = C_0\gamma_0 + r^{1/2}\sum_{k=1}^{\infty} C_k r^{z_k}\gamma_k \tag{2.2}$$

$$\gamma_0 = -4\pi G(\cos\theta_2 - \cos\theta_1)\quad(\cos^2\theta_1 + 2\nu\cos\theta_1\cos\theta_2 + \cos^2\theta_2)$$

$$\gamma_k = 4\pi G\int_{\theta_1}^{\theta_2}(Q_{rk}\cos\theta - T_k\sin\theta)\sin\theta\,d\theta \tag{2.3}$$

We shall show that all γ_k $(k = 1, 2, 3, \ldots)$ are equal to zero. Consider the following boundary value problem

$$\sigma_r = r_1^{z_k - 3/2} Q_{rs}, \quad \tau_{r\theta} = r_1^{z_k - 3/2} T_s \qquad (r = r_1)$$
$$\sigma_r = r_2^{z_k - 3/2} Q_{rs}, \quad \tau_{r\theta} = r_2^{z_k - 3/2} T_s \qquad (r = r_2) \tag{2.4}$$

Assuming that the solutions of (2.4) are unique, we obtain them by setting in (1.2) and (1.3) $C_k = 0$ for all $k \neq s$ and $C_s = 1$. The principal vector corresponding to the state of stress of the problem (2.4) has the form

$$P_s = 4\pi G r^{z_s + 1/2}\gamma_s \tag{2.5}$$

By virtue of the condition that the problem of the theory of elasticity has a solution, the vector P_s cannot depend on r, consequently $P_s = 0$ and $\gamma_s = 0$. Thus the complex zeros z_k have a corresponding state of stress which is self-equilibrating at each cross section $r = \text{const}$. We obtain the following final expression for the principal vector:

$$P = \gamma_0 C_0 \tag{2.6}$$

Further investigation will be conducted under the assumption that the difference $2\varepsilon = \theta_2 - \theta_1$ is small. Let us therefore set $\theta_1 = \theta_0 - \varepsilon$ and $\theta_2 = \theta_0 + \varepsilon$ assume that the parameter ε is small and that $0 < \xi_1 < \theta_0 < \xi_2 < 1/2\pi$ (ξ_1 and ξ_2 are some constants).

We note that the value $\theta_0 = 1/2\pi$ corresponds to a plate of variable thickness and will not be considered here.

It was shown in [2] that the roots of the characteristic equation (1.10) can be divided into three groups, according to the character of their asymptotic behavior when $\varepsilon \to 0$

$$1.\ z_{01} = -0.5, \qquad z_{02} = 0.5 \tag{2.7}$$

$$2.\ z_k = \varepsilon^{-1/2}(a_{-1k} + \varepsilon a_{1k} + \ldots) \qquad (k = 1, 2, 3, 4)$$
$$a_{-1k}^4 + 3(1 - \nu^2)\operatorname{ctg}^2\theta_0 = 0$$
$$a_{1k} = (40\, a_{-1k})^{-1}[24(1 - \nu^2)\operatorname{ctg}^2\theta_0 + 5(9 - 8\nu)] \tag{2.8}$$

$$3.\ z_k = \varepsilon^{-1}[b_{-1k} + o(\varepsilon^2)] \qquad (k = 5, 6, \ldots) \tag{2.9}$$
$$\sin^2 2b_{-1k} - 4b_{-1k}^2 = 0$$

We set for convenience $a_{-1k} = \alpha_k$, $a_{1k} = \beta_k$ and $b_{-1k} = \delta_l$ $(l = k - 4)$. The solution (1.4) and (1.5) corresponds to the roots of the first group. It has been shown that this solution can be used to remove the principal stress vector from the end faces. We note that the root $z_{02} = 0.5$ corresponds to translation of the cone as a rigid body.

Below we shall show that the zeros of the second and third group correspond to tne solutions of the edge effect with varying stress-strain state indices.

Let us transform the solution (1.2), (1.3) taking into account the smallness of ε and the formulas (2.7) to (2.9). Setting $\theta = \theta_0 + \varepsilon\eta$ and $r = r_1\rho$, $-1 \leqslant \eta \leqslant 1$ and expanding all expressions contained in (1.4) - (1.7) in terms of ε in accordance with the groups of zeros given above, we obtain

$$\begin{gathered} u_r = u_r^{(0)} + u_r^{(1)} + u_r^{(2)}, \qquad u_\theta = u_\theta^{(0)} + u_\theta^{(1)} + u_\theta^{(2)} \\ \sigma_r = \sigma_r^{(0)} + \sigma_r^{(1)} + \sigma_r^{(2)}, \qquad \sigma_\theta = \sigma_\theta^{(0)} + \sigma_\theta^{(1)} + \sigma_\theta^{(2)} \\ \sigma_\varphi = \sigma_\varphi^{(0)} + \sigma_\varphi^{(1)} + \sigma_\varphi^{(2)}, \qquad \tau_{r\theta} = \tau_{r\theta}^{(0)} + \tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)} \end{gathered} \tag{2.10}$$

where

$$\begin{gathered} u_r^{(0)} = r_1C_0\rho^{-1}\left[-2\cos\theta_0 + 4(1-\nu)\eta\varepsilon\sin\theta_0 + o(\varepsilon^2)\right] - A\left[\cos\theta_0 - \varepsilon\eta\sin\theta_0 + o(\varepsilon^2)\right] \\ u_\theta^{(0)} = r_1C_0\rho^{-1}\left[-2(1-\nu)\sin\theta_0 - 2(2-\nu)\varepsilon\eta\cos\theta_0 + o(\varepsilon^2)\right] + A\left[\sin\theta_0 + \varepsilon\eta\cos\theta_0 + o(\varepsilon^2)\right] \\ \sigma_r^{(0)} = C_0G\rho^{-2}\left[4(1+\nu)\cos\theta_0 - 4(2-\nu)\varepsilon\eta\sin\theta_0 + o(\varepsilon^2)\right] \\ \sigma_\theta^{(0)} = C_0G\rho^{-2}\varepsilon^2\left[2(1-2\nu)(\eta^2-1)\cos\theta_0 + o(\varepsilon^3)\right] \\ \sigma_\varphi^{(0)} = -C_0G\rho^{-2}\left[8(1-2\nu)\cos\theta_0\sin^{-2}\theta_0 - 4(1-2\nu)\varepsilon\eta(1+\cos^2\theta_0)^1\sin^{-3}\theta_0 + o(\varepsilon^2)\right] \\ \tau_{r\theta}^{(0)} = C_0G\rho^{-2}\varepsilon^2\left[2(1-2\nu)(\eta^2-1)\sin\theta_0 + o(\varepsilon)\right] \end{gathered} \tag{2.11}$$

$$\begin{gathered} u_r^{(1)} = r_1(\varepsilon/\rho)^{1/2}\sum_{k=1}^{4} A_kU_{rk}^{(1)}, \qquad u_\theta^{(1)} = r_1\varepsilon\rho^{-1/2}\sum_{k=1}^{4} A_kU_{\theta k}^{(1)} \\ \sigma_r^{(1)} = G\rho^{-3/2}\sum_{k=1}^{4} A_k\sigma_{rk}^{(1)}, \qquad \sigma_\theta^{(1)} = G\varepsilon\rho^{-3/2}\sum_{k=1}^{4} A_k\sigma_{\theta k}^{(1)} \\ \sigma_\varphi^{(1)} = G\rho^{-3/2}\sum_{k=1}^{4} A_k\sigma_{\varphi k}^{(1)}, \qquad \tau_{r\theta}^{(1)} = G\rho^{-1}(\varepsilon/\rho)^{1/2}\sum_{k=1}^{4} A_k\tau_{r\theta k}^{(1)} \end{gathered} \tag{2.12}$$

$$\begin{gathered} U_{rk}^{(1)} = \langle -12(1-\nu^2)(\eta\alpha_k + \nu\alpha_k^{-1}\operatorname{ctg}\theta_0)\operatorname{ctg}\theta_0 + \varepsilon^{1/2}[2(\nu-2)\alpha_k^2\beta_k - \\ - 3(1-\nu^2)\eta(2\alpha_k\beta_k - 3)\operatorname{ctg}\theta_0 - 6\nu(1-\nu^2)\alpha_k^{-1}\beta_k\operatorname{ctg}^2\theta_0] + \\ + \varepsilon\{2(7\nu-2)\alpha_k^2\beta_k - 42\eta(1-\nu^2)\beta_k\operatorname{ctg}\theta_0 - 3(1-\nu^2)\alpha_k^{-1}[(4\nu-5)\eta - \\ - 6\nu\operatorname{ctg}\theta_0]\operatorname{ctg}\theta_0 - 6(1-\nu^2)\alpha_k\operatorname{ctg}\theta_0[1 - 4/3(\nu+2)\operatorname{ctg}\theta_0 - \\ - (\eta-1)(\nu\eta - \nu + 2)\operatorname{ctg}\theta_0]\} + \ldots\rangle\exp(\varepsilon^{-1/2}\alpha_k\ln\rho) \end{gathered}$$

$$\begin{gathered} U_{\theta k}^{(1)} = \langle 12(1-\nu^2)\operatorname{ctg}\theta_0 + \varepsilon\{48(1-\nu^2)\alpha_k^{-1}\beta_k\operatorname{ctg}\theta_0 - [16(1-\nu^2)\operatorname{ctg}\theta_0 - \\ - 6(1+\nu)\ \nu(\eta^2-1)\operatorname{ctg}\theta_0 + (4\nu-5)\operatorname{tg}\theta_0]\alpha_k^2 + 12(1-\nu^2)\beta_k\operatorname{ctg}\theta_0 - \\ - 6(1-\nu^2)(2\nu\eta + 1 - \nu)\operatorname{ctg}^2\theta_0\} + \varepsilon^{3/2}\{6(1-\nu^2)\beta_k[1 - (2\nu\eta + 1 - \nu)\operatorname{ctg}\theta_0]\operatorname{ctg}\theta_0 + \\ + 48(1-\nu^2)\alpha_k^{-1}\beta^2\operatorname{ctg}\theta_0 - [16(1-\nu^2)\operatorname{ctg}\theta_0 - 6\nu(1+\nu)(\eta^2-1)\operatorname{ctg}\theta_0 + \\ + (4\nu-5)\operatorname{tg}\theta_0]\alpha_k^2\beta_k + 2(1+\nu)(1-2\nu)(\eta^2-1)\alpha_k\operatorname{ctg}\theta_0\} + \ldots\rangle\exp(\varepsilon^{-1/2}\alpha_k\ln\rho) \end{gathered}$$

$$\begin{gathered} \sigma_{rk}^{(1)} = 4(1+\nu)G\operatorname{ctg}\theta_0\langle -6\eta\alpha^2 + 6\varepsilon^{1/2}[\eta(2-\nu-\alpha_k\beta_k)\alpha_k + \\ + (1-\nu^2)\alpha^{-1}\operatorname{ctg}\theta_0] + \varepsilon\{6(2\eta - \nu\eta - 6)\alpha_k\beta_k - 3(1-2\nu)\eta - \\ - 3\eta\alpha_k^2\beta_k + [\operatorname{tg}\theta_0 + (3\nu\eta^2 - 6\eta\nu + 2 - 6\eta - \nu)\operatorname{ctg}\theta_0]\alpha_k^2\} + \ldots\rangle\exp(\varepsilon^{-1/2}\alpha_k\ln\rho) \end{gathered}$$

$$\begin{gathered} \sigma_{\theta k}^{(1)} = 3(1+\nu)(\eta-1)G\operatorname{ctg}\theta_0\langle 4(\eta+1)\operatorname{ctg}\theta_0[3(1-\nu^2)\operatorname{ctg}\theta_0 - 1/3(2\nu-1) + \\ + 1/3\,\nu\alpha_k^2]\alpha_k + \varepsilon^{1/2}\{4(\eta+1)\operatorname{ctg}\theta_0[3(1-\nu^2)\beta_k\operatorname{ctg}\theta_0 + 1/3\,\nu\alpha_k^2\beta_k - 1/3(2\nu-1)\beta_k]\} + \\ + \varepsilon\{32(1-\nu^2)(3\eta+2)\alpha_k^{-1}\beta_k\operatorname{ctg}^2\theta_0 - 1/2[8(1-\nu)(19\nu-7)\operatorname{ctg}^2\theta_0 + (88 - 63\nu - \\ - 20\nu^2](\eta+1)\operatorname{ctg}\theta_0 + 2/3\,\nu(\eta+1)\alpha_k^2\beta_k\operatorname{ctg}\theta_0 + \\ + \langle 4\nu(\eta+1)(\eta-6) - (3\eta+1)(16\nu-19) + \operatorname{ctg}^2\theta_0[2(9\nu^2+4\nu+ \\ + 15) + 4\nu(2\eta^2-\eta+1) - 4\nu(1-\nu)(\eta-1)^2 - 6(1-2\nu)(\eta+1)]\rangle\alpha_k^2\} + \ldots\rangle\exp(\varepsilon^{-1/2}\alpha_k\ln\rho) \end{gathered} \tag{2.13}$$

$$\sigma_{\varphi k}^{(1)} = 2G \langle 12(1+\nu)[(1-\nu^2)\operatorname{ctg}\theta_0 - \nu\eta\alpha_k^2]\operatorname{ctg}\theta_0 + \varepsilon^{1/2}\{12(1+\nu) \times$$
$$\times \alpha_k[(1-\nu^2)\operatorname{ctg}\theta_0 - \nu\eta\alpha_k^2]\operatorname{ctg}\theta_0 - 12(1+\nu)(1-2\nu)\eta\alpha_k\operatorname{ctg}\theta_0\} +$$
$$+\varepsilon\{48(1+\nu)(1-\nu^2)\alpha_k^{-1}\beta_k\operatorname{ctg}^2\theta_0 + (1+\nu)[1-8(1-2\nu)\operatorname{ctg}^2\theta_0 +$$
$$+6\nu(1+\nu)(\eta^2-1)\operatorname{ctg}^3\theta_0]\alpha_k^2 + 12(1+\nu)\alpha_k\beta_k(2\eta-\nu\eta+2\nu)\times$$
$$\times\operatorname{ctg}\theta_0 + 6(1-\nu^2)(1+\nu)\alpha_k\operatorname{ctg}^2\theta_0 - 6\nu(1+\nu)\eta\alpha_k^2\beta_k\operatorname{ctg}\theta_0 -$$
$$-3(1+\nu)[4(1-\nu^2)(\eta+1)\operatorname{ctg}^2\theta_0 + (1-\nu)(2\nu-7)]\operatorname{ctg}\theta_0\} + \ldots\rangle \exp(\varepsilon^{-1/2}\alpha_k \ln\rho)$$

$$\tau_{r\theta k}^{(1)} = 6G(1+\nu)\alpha_k(\eta^2-1)\operatorname{ctg}\theta_0\langle 2\alpha_k^2 + \varepsilon^{1/2}(2\alpha_k\beta_k - 3)\alpha_k +$$
$$+\varepsilon]11\alpha_k\beta_k + (\alpha_k - {}^1/_3\,\eta\operatorname{ctg}\theta_0)\alpha_k^2 + 6\nu - 5] + \ldots\rangle\exp(\varepsilon^{-1/2}\alpha_k\ln\rho)$$

$$u_r^{(2)} = r_1\varepsilon\rho^{-1/2}\sum_{l=0}^{\infty} B_l[(1-k)\delta_l F_l(\eta) + (1+k)\delta_l^{-1}F_l''(\eta)]\exp(\varepsilon^{-1}\delta_l\ln\rho)$$

$$u_\theta^{(2)} = -r_1\varepsilon\rho^{-1/2}\sum_{l=1}^{\infty} B_l[(1+3k)F_l'(\eta) + (1+k)\delta_l^{-2}F''(\eta)]\exp(\varepsilon^{-1}\delta_l\ln\rho)$$

$$\sigma_r^{(2)} = 2G\rho^{-3/2}\sum_{l=1}^{\infty} B_l F_l''(\eta)\exp(\varepsilon^{-1}\delta_l\ln\rho)$$

$$\sigma_\theta^{(2)} = 2G\rho^{-3/2}\sum_{l=1}^{\infty} B_l\delta_l^2 F_l(\eta)\exp(\varepsilon^{-1}\delta_l\ln\rho) \tag{2.14}$$

$$\sigma_\varphi^{(2)} = 2G\rho^{-3/2}\sum_{l=1}^{\infty} B_l[F_l''(\eta) + \delta_l^2 F_l(\eta)]\exp(\varepsilon^{-1}\delta_l\ln\rho)$$

$$\tau_{r\theta}^{(2)} = -2G\rho^{-3/2}\sum_{l=1}^{\infty} B_l F_l'(\eta)\exp(\varepsilon^{-1}\delta_l\ln\rho)$$

Here A_k and B_l are new unknown constants and $k = (1-2\nu)^{-1}$; $F_l(\eta)$ in (2.14) denote the Papkovich functions,

$$F_l(\eta) = (\delta_l^{-1}\sin\delta_l + \cos\delta_l)\cos\delta_l\eta + \eta\sin\delta_l\sin\delta_l\eta \qquad (l = 1, 3, \ldots) \tag{2.15}$$

$$F_l(\eta) = (\sin\delta_l - \delta_l^{-1}\cos\delta_l)\sin\delta_l\eta + \eta\cos\delta_l\cos\delta_l\eta \qquad (l = 2, 4, \ldots) \tag{2.16}$$

3. We consider the problem of removing the stresses from the end surfaces of the shell. Let the following stresses be prescribed when $r = r_s$ $(s = 1, 2)$

$$\sigma_r = f_{1s}(\theta), \qquad \tau_{r\theta} = f_{2s}(\theta) \tag{3.1}$$

Functions $f_{js}(\theta)$ satisfy the conditions of equilibrium

$$2\pi r_1^2\int_{\theta_1}^{\theta_2}(f_{11}\cos\theta - f_{21}\sin\theta)\sin\theta d\theta = 2\pi r_2^2\int_{\theta_1}^{\theta_2}(f_{12}\cos\theta - f_{22}\sin\theta)\sin\theta d\theta = P \tag{3.2}$$

where P is the principal stress vector acting in any cross section $r = \text{const}$.

As we have shown before, the non-self-equilibrating part of the stresses (3.1) can be removed using the penetrating solution (1.4) and (1.5) with the constant C_0 and the principal stress vector P connected by Eq. (2.6). Below we shall assume that $P = 0$.

We shall seek the solution in the form (1.2), (1.3). According to the assumption made above that $C_0 = 0$. The arbitrary constants C_k whose variations are assumed independent, will be determined using as in [3, 4], the Lagrange's variational principle.

Since the homogeneous solutions satisfy the equations of equilibrium and the boundary conditions on the conical surface, the variational principle assumes the following form

$$r_1 \sum_{s=1}^{2} \rho_s^2 \int_{\theta_1}^{\theta_2} [(\sigma_r - f_{1s})\, \delta u_r + (\tau_{r\theta} - f_{2s})\, \delta u_\theta]_{\rho=\rho_s} \sin\theta\, d\theta = 0 \qquad (3.3)$$

Equating to zero the coefficients accompanying the independent variations δC_k, we obtain the following infinite system:

$$\sum_{k=1}^{\infty} m_{jk} C_k = a_j \qquad (j = 1, 2, 3, \ldots) \qquad (3.4)$$

where

$$m_{jk} = (\rho_1^{z_j + z_k} + \rho_2^{z_j + z_k}) \int_{\theta_1}^{\theta_2} (Q_{rk} U_{rj} + T_k U_{\theta j}) \sin\theta\, d\theta \qquad (3.5)$$

$$a_j = \sum_{s=1}^{2} \rho_s^{z_j + 3/2} \int_{\theta_1}^{\theta_2} (f_{1s} U_{rj} + f_{2s} U_{\theta j}) \sin\theta\, d\theta \qquad (3.6)$$

It can be shown that this system is positive definite in the energy space $H_E^{'}$ and therefore has a solution whenever physically meaningful conditions are imposed on its right-hand side.

Using the smallness of the shell thickness parameter $2\varepsilon = \theta_2 - \theta_1$, we can construct an asymptotic solution of (3.4). We begin by sharpening the assumptions concerning the external load.

Assume that $f_{1s} \sim 1$. Then the assumption that σ_r and $\tau_{r\theta}$ corresponding to the roots of the second group are of different order $(\sigma_r^{(1)} \sim 1,\ \tau_{r\theta}^{(1)} \sim \sqrt{\varepsilon})$ implies that the choice of the order of f_{2s} must be guided by the following considerations. Using the formulas (2.13) and (2.14) and the fact that $F_k(\pm 1) = 0$, we obtain

$$\int_{-1}^{1} \tau_{r\theta}\, d\eta = -16\, G\, (1 + \nu)\, \rho^{-3/2} \operatorname{ctg} \theta_0 \varepsilon^{1/2} \sum_{k=1}^{4} A_k \alpha_k^3 \exp(\varepsilon^{-1/2} \alpha_k \ln\rho) \qquad (3.7)$$

Writing now the tangential stresses prescribed at the boundary in the form

$$f_{2s} = f_{2s}^{(1)} + f_{2s}^{(2)}, \qquad f_{2s}^{(1)} = \int_{\theta_1}^{\theta_2} f_{2s}\, d\eta, \qquad f_{2s}^{(2)} = f_{2s} - f_{2s}^{(1)} \qquad (3.8)$$

we find that the asymptotic formula (3.7) leads us to the necessary assumption that $f_{2s}^{(1)}$ is of the order of $\varepsilon^{1/2}$, while $f_{2s}^{(2)}$ may be of the same order as f_{1s}, i.e. $f_{2s}^{(2)} \sim 1$.

Further, using the formulas (2.13) and (2.14) we seek the constants A_k and B_l in the form

$$A_k = A_{k0} + \sqrt{\varepsilon}\, A_{k1} + \ldots, \qquad B_l = B_{l0} + \sqrt{\varepsilon}\, B_{l1} \qquad (3.9)$$

Taking into account the order of the stresses prescribed at the boundary, we now use the variational principle to obtain the following system of equations for A_{k0} and B_{l0}:

$$\sum_{k=1}^{4} n_{jk} A_{k0} = a_j \qquad (j = 1, 2, 3, 4 \ldots) \qquad (3.10)$$

$$\sum_{l=1,3}^{\infty} g_{tl}B_{l0} = b_t \quad (t=1,3,\ldots), \qquad \sum_{l=2,4}^{\infty} g_{tl}B_{l0} = b_t \quad (t=2,4,\ldots) \tag{3.11}$$

where

$$n_{jk} = 16\,G\,(1-\nu^2)\,\alpha_k^2(\alpha_j - d_k)\operatorname{ctg}^2\theta_0 \tag{3.12}$$

$$a_j = \sum_{s=1}^{2} \rho_s^{1/2} \exp(\varepsilon^{-1/2}\alpha_j \ln\rho_s) \int_{-1}^{1} [f_{2s} - f_{1s}\varepsilon^{-1/2}(\eta\alpha_j + \nu\operatorname{ctg}\theta_0\,\alpha_j^{-1})]\,d\eta$$

$$g_{tl} = 4G\frac{\delta_t^2\delta_l^2(\sin^2\delta_t - \sin^2\delta_l)}{(\delta_t^2-\delta_l^2)(\delta_t-\delta_l)}[(k-1)(\delta_t^2+\delta_l^2)+$$

$$+2(k+1)\delta_t\delta_l]\sum_{s=1}^{2}\rho_s^2\exp\left(\frac{\delta_t+\delta_l}{\varepsilon}\ln\rho_s\right)$$

$$g_{tt} = 4G\delta_t^2(1 - {}^2/_3\sin^2\delta_t)\sum_{s=1}^{2}\rho_s^2\exp(2\delta_t/\varepsilon\ln\rho_s)$$

$$b_t = \sum_{s=1}^{2}\rho_s^{3/2}\exp\left(\frac{\delta_t}{\varepsilon}\ln\rho_s\right)\int_{-1}^{1}\left\{f_{1s}\left[(1-k)\,\delta_t F_t(\eta) + (1+k)\frac{F_t''}{\delta_t^2} - \right.\right.$$

$$\left.\left. - f_{2s}\left[(1+3k)\,F_t' + (1+k)\frac{F_t''}{\delta_t^2}\right]\right\}d\eta \qquad (t,\,l = 1,3,\ldots)$$

For $t,\,l = 2,4,\ldots$ the corresponding expressions for g_{tl} are obtained when replacing in the above formulas $\cos\delta_l$ by $\sin\delta_l$ and $\sin\delta_l$ by $-\cos\delta_l$ respectively.

The structure of the system obtained enables us to conclude that the unknowns A_{k0} corresponding to the second group of zeros and the unknowns B_{k0} corresponding to the third group of zeros, can be obtained independently.

The process of determining A_{ki} and B_{ki} $(i = 1, 2,\ldots)$ can invariably be reduced to inverting identical matrices coinciding with the matrices of (3.10) and (3.11).

It should be noted that the systems (3.11) have been already encountered in the theory of thick plates [5, 6] and served repeatedly as the basis of numerical analysis of various problems.

The system (3.10), (3.11) becomes considerably simplified when the state of stress of a semi-infinite cone $(\rho_1 = 1,\ \rho_2 \to \infty)$ or of a cone with an apex $(\rho_2 = 1,\ \ \rho_1 = 0)$. is considered.

In the first case all unknowns corresponding to the zeros for which $\operatorname{Re}\alpha_k > 0$ and $\operatorname{Re}\delta_l > 0$, should be made equal to zero. In the second case the boundedness of the solution at the apex suggests equating to zero those unknowns, for which $\operatorname{Re}\alpha_k < 0$ and $\operatorname{Re}\delta_l < 0$. Both cases yield systems which are formally identical

$$\sum_{k=1}^{2} n_{jk}^{(0)}A_{k0} = a_j^{(0)} \quad (j=1,2) \tag{3.13}$$

$$\sum_{l=1}^{\infty} g_{tl}^{(0)}B_{l0} = b_t^{(0)} \quad (t=1,2,\ldots)$$

The coefficients and the right-hand sides of (3.13) are easily obtained from the corresponding coefficients and right-hand sides of (3.10) and (3.11) by setting $\rho_1 = 1$ and

$\rho_2 \to \infty$ in the first case, and $\rho_1 = 0$ and $\rho_2 = 1$ in the second case.

Let us now clarify the pattern of the state of stress corresponding to the zeros of the second and the third group. We set, for simplicity, $\rho_1 = 0$ and $\rho_2 = 1$ ($r = \rho r_2$) and compute the bending moment and the shearing force for each group of solutions. We have

$$M = r_2^2 \int_{\theta_1}^{\theta_2} \{\sigma_r \sin(\theta - \theta_0) - \tau_{r\theta}[1 - \cos(\theta - \theta_0)]\} \sin\theta d\theta \approx$$

$$\approx \varepsilon^2 r_2^2 \sin\theta_0 \int_{-1}^{1} \sigma_r dr + o(\varepsilon^3) \tag{3.14}$$

$$Q = r_2^2 \int_{\theta_1}^{\theta_2} [\sigma_r \sin(\theta - \theta_0) + \tau_{r\theta}\cos(\theta - \theta_0)] \sin\theta d\theta = \varepsilon r_2 \sin\theta_0 \int_{-1}^{1} \tau_{r\theta} d\eta + o(\varepsilon^2)$$

Inserting now the expressions for the stresses, we obtain

$$M_1 = -32G(1+\nu)\varepsilon^2\rho^{-3/2} r_2^2 \cos\theta_0 \sum_{k=1}^{4} A_{k0}\alpha_k^2 \exp(\varepsilon^{-1/2}\alpha_k \ln\rho) + o(\varepsilon^{5/2})$$

$$Q_1 = -8G(1+\nu) r_2 \cos\theta_0 \left(\frac{\varepsilon}{\rho}\right)^{3/2} \sum_{k=1}^{4} A_k\alpha_k^3 \exp(\varepsilon^{-1/2}\alpha_k \ln\rho) + o(\varepsilon^2) \tag{3.15}$$

$$M_2 = o(\varepsilon^3), \qquad Q_2 = o(\varepsilon^2) \tag{3.16}$$

Thus the principal parts of the bending moment and the shearing force determine the solution for the second group.

In conclusion we note that the asymptotic method developed in this paper can be used to remove the stresses from the end faces of the boundary of a conical shell. From the conical part of the boundary the stresses can be removed by constructing applied theories with help of the methods and examples given in [3, 4], and this alone could merit a special study. The removal could be realized by solving the problem of the theory of elasticity for an infinite conical shell with the help of the Mellin transformation.

BIBLIOGRAPHY

1. Nuller, B. M., On solving the problem of the theory of elasticity for a truncated hollow cone. Inzh. zh., MTT, №5, 1967.
2. Mekhtiev, M. F. and Ustinov, Iu. A., Asymptotic behavior of the solution of the axisymmetric problem of the theory of elasticity for a hollow cone. Proc. of the 7-th All-Union Conference on the Theory of Shells and Plates., Dnepropetrovsk, 1969, M., "Nauka" 1969.
3. Vilenskaia, T. V. and Vorovich, I. I., Asymptotic behavior of the solution of the problem of the theory of elasticity for a thin spherical shell. PMM Vol. 30, №2, 1966.
4. Bazarenko, N. A. and Vorovich, I. I., Asymptotic solution of the elasticity problem for a hollow, finite length thin cylinder. PMM Vol. 29, №6, 1965.
5. Aksentian, O. K. and Vorovich, I. I., The state of stress in a thin plate. PMM Vol. 27, №6, 1963.
6. Vorovich, I. I. and Malkina, O. S., The state of stress in a thick plate. PMM Vol. 31, №2, 1967.

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